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Open Problems Session:

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Title: *Geometry of integrable systems defined by curves other than hyper-elliptic ones*

Abstract: In this short presentation I will address the issue of finding relevant geometric framework for integrable systems generated by separation curves other than hyper-elliptic ones.

Every smooth n-parameter curve in the (λ, μ) plane

$$\phi(\lambda, \mu, h_1, \ldots, h_n) = 0 \tag{1}$$

leads to a Liouville integrable and separable finite-dimensional Hamiltonian system with n Hamiltonians in involution

$$h_i = h_i(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n), \quad i = 1, \ldots, n, \qquad \{h_i, h_j\} = 0 \tag{2}$$

The vast majority of research focuses on curves of hyper-elliptic type:

$$\sigma(\lambda) + \sum_{i=1}^{n} h_i \lambda^{\gamma_i} = f(\lambda)\mu^2 \tag{3}$$

where $\gamma_i \in \mathbb{Z}$, σ and f are polynomials, that generate quadratic in momenta separable integrable systems for which there exist a natural geometric interpretation. The corresponding Hamiltonians constitute an integrable system in a pseudo-riemannian space, and a subalgebra of Killing tensors that spans the separation web exists. A sub-class with flat metrics is recognized, and flat coordinates are found. Multi-hamiltonian structure with corresponding Miura-type maps and Lax formulation are well understood.

Virtually nothing is known, however, if we leave this class of curves and pass to higher order algebraic case, apart from some specific cases (see for example the curve for the stationary Bousinessq system in [3] that is third order in μ). No natural geometry connected

to such systems is known. Is there any way of relating a pseudo-riemannian metric to such systems? Is there any canonical (albeit non pointwise) transformation of phase-space variables that turn these systems into systems with some of the Hamiltonians quadratic in momenta? Are there any Killing tensors related to such systems? How to find Lax formulation for such systems?

References

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