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Open Problems Session:

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Title: *Open problem: discretize Gauss-Codazzi equations*

Abstract: The moving frame of surfaces in the usual space $\mathbb{R}^3$ (Bobenko 1994)

$$\mathbb{U} = \begin{pmatrix} (1/4)u_z & -Qe^{-u/2} \\ (1/2)(H+c)e^{u/2} & -(1/4)u_z \end{pmatrix}, \mathbb{V} = \begin{pmatrix} -(1/4)u_{\bar{z}} & -(1/2)(H-c)e^{u/2} \\ \overline{Q}e^{-u/2} & (1/4)u_{\bar{z}} \end{pmatrix}, \tag{1}$$

generates by the zero-curvature condition

$$[\partial_z - \mathbb{U}, \partial_{\bar{z}} - \mathbb{V}] = \mathbb{U}_{\bar{z}} - \mathbb{V}_z + [\mathbb{U}, \mathbb{V}[= 0, \tag{2}$$

three PDEs in four fields (u and H real, Q and $\overline{Q}$ complex conjugate), the Gauss-Codazzi equations

$$\begin{cases} u_{z\bar{z}} + \frac{1}{2}H^2 e^u - 2|Q|^2 e^{-u} = 0 \text{ (Gauss)}, \\ Q_{\bar{z}} - \frac{1}{2}H_z e^u = 0,\ \overline{Q}_z - \frac{1}{2}H_{\bar{z}} e^u = 0 \text{ (Codazzi)}. \end{cases} \tag{3}$$

Open problem. Discretize (d- or q-) either the linear system (1) or the nonlinear system (3) and, of course, require their continuum limit to be the continuous equations.

There exist many partial results (Bobenko, Konopelchenko, Nijhoff, Schief, ...), but not this general one.

Motivation. Since the moving frame of Bonnet surfaces is equivalent to the best matrix Lax pair of continuous P_{VI} (RC 2017), a by-product should be the best discrete matrix Lax pair of possibly the best discrete P_{VI}.

References

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